From old to recent paradoxes in fluid mechanics

(and how to overcome the lubrication one)

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Outline

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6. Two examples
7. Conclusions

Isothermal fluid dynamics equations

Define $\boldsymbol{v}(\boldsymbol{x},t)$ to be the **eulerian particle velocity**, $\varrho(\boldsymbol{x},t)$ the **fluid density**, $\mathbb{T}(\boldsymbol{x},t)$ the (symmetric) **Cauchy stress tensor**, $\boldsymbol{f}(\boldsymbol{x},t)$ the **body force** (e.g. gravity), ω_t a **material fluid volume** with outer unit normal $\boldsymbol{n}$

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$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_{\omega_t} \varrho = 0 \qquad \text{(mass conservation principle)}$$

Isothermal fluid dynamics equations (local form)

Under regularity conditions (on both motion and fluid domain)

$$(1)\qquad \begin{cases} \varrho\left(\dfrac{\partial}{\partial t}\boldsymbol{v}) + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\right) = \boldsymbol{f} + \nabla\cdot\mathbb{T} & \text{(Navier-Stokes eq.)} \\ \mathbb{T} = \mathbb{T}^{\mathrm{T}} & \text{(symmetry)} \\ \dfrac{\partial\,\varrho}{\partial t} + \nabla\cdot(\varrho\boldsymbol{v}) = 0 & \text{(continuity eq.)} \end{cases}$$

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So, in special problems, various approximations/simplifications are applied

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For those that are not in the "fluid dynamics business" (or are just simply curious) may be interesting a quick look at the most famous ones

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Stationary **inviscid** ($\mathbb{T} = -p\mathbb{I}$) potential flow past a finite obstacle Ω

$$\begin{cases} \varrho \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p, \qquad \boldsymbol{v} = \nabla \phi, \\ \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega} = 0, \qquad \boldsymbol{v} \to \boldsymbol{U}_{\infty}, \quad \text{for } |\boldsymbol{x}| \to \infty \end{cases}$$

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This is the celebrated **Boundary layer theory** (Prandtl, 1918), very well known to engineers but whose mathematical consistency was proved only fifty years later (Olienik, Serrin, Fife...) **(although other important mathematical problems remain still open!)**

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Why? Wrong linearization approach?

NO! A more sophisticated linearization procedure for the fluid motion past a sphere (three dimensional case) shows the same inconsistency. This is the Whitehead's paradox (1889)

1910: Oseen’s approach

***Suggestion*: inertia forces can be neglected far from the obstacle but not close to it**

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Oseen approach to Stokes' paradox: linearize the momentum equation around the velocity at infinity

$$\boldsymbol{U}_\infty \cdot \nabla \boldsymbol{v} = -\nabla p + \eta \nabla^2 \boldsymbol{v}$$

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Consequence

Even the Oseen equation does not represent a valid approximation of the full Navier-Stokes equations which remains valid uniformly in the flow domain.

Other (less known) examples

- **Jeffrey-Hamel's paradox** (J. Fluid Mechanics 1980) and **Moffatt's solution** (J. Fluid Mechanics 1980, *Local similarity solutions and their limitations*)
- **Gray's paradox** (J. Exp. Biol 1936, *Studies in animal locomotion: the propulsive powers of the dolphin*) and **Bale's and al. solution** (Nature 2014, *Gray's paradox: a fluid mechanical perspective*)
- **Muller's paradox** (Thermodynamics, 1980) and **Gouin's and al. solution** (Continuum Mech. Thermodyn. 2012, *On the Müller's paradox for thermal incompressible media*)

Visco-plastic fluids

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These fluids can flow (i.e., deform indefinitely) only if they are submitted to a stress above some critical value. Below this threshold they remain rigid or “almost” rigid.

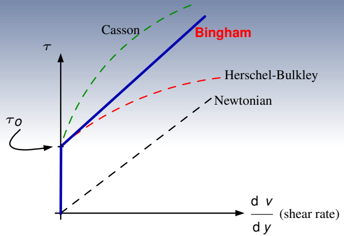
Visco-plastic (incompressible) fluids in “simple” shear rate regime

Laminar (simple-shear) flow: $\boldsymbol{v} = (v(y), 0, 0)$, (or $\boldsymbol{v} = (0, v(r), 0)$ in a rotational geometry), $\mathbb{S} = \mathbb{T} + P\mathbb{I}$ (shear stress), being $P \overset{\text{def}}{=} -\dfrac{1}{3}(\operatorname{tr}\mathbb{T})$ (mechanical pressure),

$$\mathbb{S} = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & \end{pmatrix}$$

where $\tau = \tau(dv/dy)$ (or $\tau = \tau(dv/dr)$). Using a viscometer rheologists classify fluid behaviors, i.e. identify τ, the unique non-zero component of the shear stress as a function of the shear rate (the spatial derivative of v)

Examples of 1D rheograms



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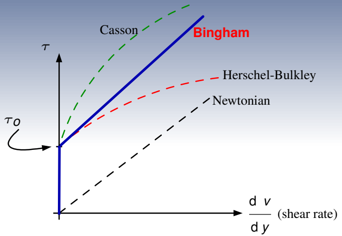

Visco-plastic fluids are characterized by a yield stress τ_o below which they do not deform ($dv/dy = 0$) and above which they flow according to different constitutive relations.

- $\tau = \eta \dfrac{\mathrm{d}\,v}{\mathrm{d}\,y}$ (Newtonian: no yield stress)
- $\boxed{\tau = \tau_o + \eta \dfrac{\mathrm{d}\,v}{\mathrm{d}\,y}}$ (Bingham)
- $\tau = \tau_o + \eta \left(\dfrac{\mathrm{d}\,v}{\mathrm{d}\,y}\right)^n$ (Herschel-Bulkley)
- $\sqrt{\tau} = \sqrt{\tau_o} + \sqrt{\eta \dfrac{\mathrm{d}\,v}{\mathrm{d}\,y}}$ (Casson)

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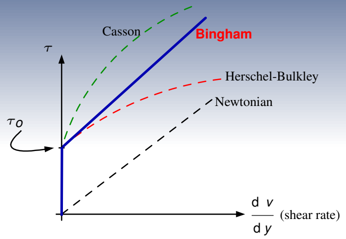

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The Bingham model is the simplest viscoplastic model. Above the yield stress the fluid behaves like a linear viscous (Newtonian) fluid. At zero shear rate, τ remains undetermined.

The Bingham constitutive model (3D)

Similar but more complicated!

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Similar but more complicated! Introduce

- $\mathbb{S}^* = \mathbb{T}^* + P^*\mathbb{I}$ (shear or "extra" stress), $P^* = -\mathrm{tr}(\mathbb{T}^*)/3$ (dynamic pressure)
- $\mathbb{D}^* = \dfrac{1}{2}\left[\nabla \boldsymbol{v}^* + (\nabla \boldsymbol{v}^*)^T\right]$ (rate of deformation or "strain rate" tensor)
- $II_{\mathbb{S}^*} = \sqrt{\dfrac{1}{2}\,\mathrm{tr}\,\left(\mathbb{S}^{*2}\right)}$, $\quad II_{\mathbb{D}^*} = \sqrt{\dfrac{1}{2}\,\mathrm{tr}\,\left(\mathbb{D}^{*2}\right)}$ (second invariant of $\mathbb{S}^*$ and $\mathbb{D}^*$ respectively)

All functions depend on $\boldsymbol{x}$, t coordinates.

Note that, by definition, $\mathbb{S}^*$ is traceless.

Starred variables are dimensional (however to work out the analysis we will use dimensionless variables).

The Bingham constitutive model (3D)

Generalization of the 1-D constitutive law

$$\begin{cases} \mathbb{S}^* = \left(\dfrac{\tau_0^*}{II_{D^*}} + 2\eta \right) \mathbb{D}^*, & \text{when} \quad II_{S^*} > \tau_0^* \\ \mathbb{D}^* = \mathbb{O} & \text{when} \quad II_{S^*} \leq \tau_0^* \end{cases}$$

The model depend on two material (positive scalar) parameters: η the **plastic viscosity**, τ_o^* the **yield stress**

When the shear stress is below τ_o^* a **solid-like** behavior is observed, when it is above a **fluid-like** behavior appears.

Recall that **in the rigid region the stress is not determined**.

The related free-boundary problem

$$\mathbb{D}^* = \mathbb{O} \qquad \text{(equation of motion in the rigid region)}$$

$$\varrho\left(\frac{\partial\, \boldsymbol{v}^*}{\partial t^*} + \boldsymbol{v}^* \cdot \nabla^* \boldsymbol{v}^*\right) = -\nabla P^* + \nabla \cdot \mathbb{S}^* + \varrho \boldsymbol{f}^* \qquad \text{(equation of motion in the fluid region)}$$

$$\mathrm{tr}\, \mathbb{D}^* = \frac{\partial v_1^*}{\partial x^*} + \frac{\partial v_2^*}{\partial y^*} + \frac{\partial v_3^*}{\partial z^*} = 0 \qquad \text{(incompressibility condition)}$$

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Transition between solid–liquid regimes depends upon being $II_{\mathbb{S}^*} \leq \boldsymbol{\tau}_o^*$ (RIGID, $\mathbb{D}^* = 0$) or $II_{\mathbb{S}^*} \geq \boldsymbol{\tau}_o^*$ (FLUID, $\mathbb{D}^* \neq 0$). Thus an **unknown** boundary separates these regimes, implicitly defined by $II_{\mathbb{S}^*} = \boldsymbol{\tau}_o^*$!

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this problem (coupled with initial, boundary and yield surface conditions) may be inconsistent for certain special (but important) geometries mostly used in the so-called "lubrication approximation"!

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$$\mathbb{D}^* = \mathbb{O} \qquad \text{(equation of motion in the rigid region)}$$

$$\varrho\left(\frac{\partial \boldsymbol{v}^*}{\partial t^*} + \boldsymbol{v}^* \cdot \nabla^* \boldsymbol{v}^*\right) = -\nabla P^* + \nabla \cdot \mathbb{S}^* + \varrho \boldsymbol{f}^* \qquad \text{(equation of motion in the fluid region)}$$

$$\mathrm{tr}\, \mathbb{D}^* = \frac{\partial v_1^*}{\partial x^*} + \frac{\partial v_2^*}{\partial y^*} + \frac{\partial v_3^*}{\partial z^*} = 0 \qquad \text{(incompressibility condition)}$$

von Mises's criterion

Transition between solid–liquid regimes depends upon being $II_{\mathbb{S}^*} \leq \boldsymbol{\tau}_o^*$ (RIGID, $\mathbb{D}^* = 0$) or $II_{\mathbb{S}^*} \geq \boldsymbol{\tau}_o^*$ (FLUID, $\mathbb{D}^* \neq 0$). Thus an **unknown** boundary separates these regimes, implicitly defined by $II_{\mathbb{S}^*} = \boldsymbol{\tau}_o^*$!

However...

this problem (coupled with initial, boundary and yield surface conditions) may be inconsistent for certain special (but important) geometries mostly used in the so-called "lubrication approximation"! A paradox again!

What do we mean by *lubrication theory*?

Assume that the fluid domain has TWO characteristic lengths, H and L such that $\varepsilon =: \dfrac{H}{L} \ll 1$ (e.g. a thin layer between parallel plates, a liquid film on a given surface, etc.)

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Define now the following *dimensionless* numbers

$$\underbrace{\mathrm{Fr}^2}_{\text{Froude n.}} =: \frac{U^2}{FL} \left(= \frac{\text{inertia force}}{\text{body force}} \right), \qquad \underbrace{\mathrm{Eu}}_{\text{Euler n.}} =: \frac{\Pi}{\varrho U^2} \left(= \frac{\text{pressure force}}{\text{inertia force}} \right),$$

$$\underbrace{\mathrm{Re}}_{\text{Reynolds n.}} =: \frac{\varrho U H}{\eta} \left(= \frac{\text{inertia force}}{\text{viscous force}} \right)$$

Π characteristic pressure, F characteristic body force, U characteristic velocity

Lubrication theory (2D case) in the fluid region

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Balance equations in dimensionless form

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0,$$

$$\varepsilon \mathrm{Re}\left(\frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x} v_1 + \frac{\partial v_1}{\partial y} v_2\right) = \varepsilon\left(\frac{\mathrm{Re}}{\mathrm{Fr}^2}\right) f_1 - \varepsilon(\mathrm{Eu}\cdot\mathrm{Re})\frac{\partial P}{\partial x} + \left[\varepsilon\frac{\partial S_{11}}{\partial x} + \frac{\partial S_{12}}{\partial y}\right],$$

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Choose convenient values of Fr, Eu, Re and look for solutions as

$$\phi(x,y,t;\varepsilon) = \phi^{(0)}(x,y,t) + \varepsilon\phi^{(1)}(x,y,t) + \varepsilon^2\phi^{(2)}(x,y,t) + \ldots\ .$$

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Analyze BVP obtained by equating same order terms in ε.

Depending on the chosen values of Fr, Eu, Re several simplified equations follows.

The lubrication paradox

On the other hand...

the model with a rigid core leads to a contradiction for complex (i.e. non-elementary) geometries of the fluid domain: *the analysis shows that the core has a non-constant velocity in the longitudinal direction, which is impossible!* [Lipscomb & Denn (JNNFM,1984)]

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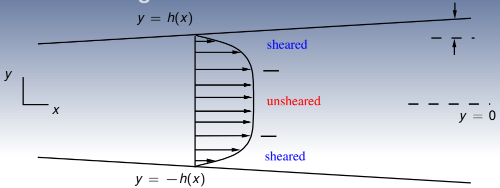
For "elementary" geometries (say laminar flow between parallel planes or in a pipe with constant cross-section) the free-boundary problem both stationary and unsteady has been fully investigated, well posedness proved and no inconsistency appears in the model

Lipscomb & Denn's argument



$x \in (0, L)$, $y \in (-h(x), h(x))$, $H/L = \varepsilon \ll 1$, $h(x) = H + \varepsilon x$

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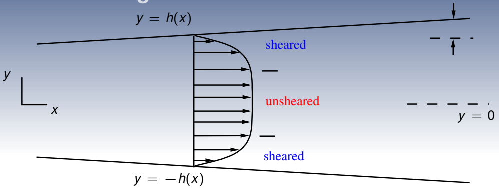

$$x \in (0, L),\, y \in (-h(x), h(x)),\, H/L = \varepsilon \ll 1,\, h(x) = H + \varepsilon x$$

Pressure driven fully developed steady Bingham flow with $\alpha = O(\varepsilon)$ (slightly divergent planes). Balance equations in the lubrication approximation show that, at the zeroth order in α

$$p = p(x), \qquad \tau_{xy} = y \frac{\mathrm{d}\, p}{\mathrm{d}\, x}, \qquad \sigma(x) = -\frac{\tau_0}{dp/dx}.$$

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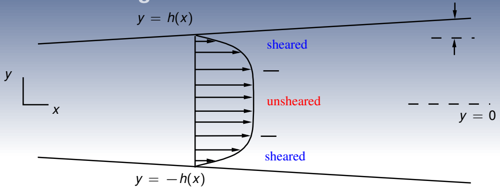

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Pressure is given at the entrance $p(x = 0) = P_{in}$ and measured at the exit of the channel ($p(x = L) = P_{out}$), so that $(P_{in} - P_{out})/L < 0$ is the assigned pressure gradient,

but $p(x)$ is unknown in $(0, L)$ and must be determined.

Lipscomb & Denn's argument

Longitudinal velocity field as a function of $p'(x)$

(2) $$v_x = \frac{1}{2\eta}\frac{\mathrm{d}\,p}{\mathrm{d}\,x}\left(|y|^2 - h^2(x)\right) + \frac{\tau_0}{\eta}(|y| - h(x)), \qquad \sigma(x) \leq |y| \leq h(x)$$

(3) $$v_x = \frac{1}{2\eta}\frac{\mathrm{d}\,p}{\mathrm{d}\,x}\left(\sigma^2(x) - h^2(x)\right) + \frac{\tau_0}{\eta}(\sigma(x) - h(x)), \quad |y| \leq \sigma(x)\left(= -\frac{\tau_0}{dp/dx}\right)$$

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Impose a **constant** flow rate (per unit width) $Q = \int_{-h(x)}^{h(x)} v_x dy$: this implies that $p(x)$ must obey the ODE

(4) $$Q = \frac{2h^2(x)\tau_0}{3\eta}\left[-\frac{h(x)}{\tau_0}\frac{\mathrm{d}\,p}{\mathrm{d}\,x}\right] \times \left\{1 - \frac{3}{2}\left[-\frac{h(x)}{\tau_0}\frac{\mathrm{d}\,p}{\mathrm{d}\,x}\right]^{-1} + \frac{1}{2}\left[-\frac{h(x)}{\tau_0}\frac{\mathrm{d}\,p}{\mathrm{d}\,x}\right]^{-3}\right\}$$

(The latter applies only when $\sigma < h$; if $\sigma \ge h$ there is no flow)

Lipscomb & Denn's argument

Equation (4) **cannot have a solution of type** $p'(x)$ =**constant**. Indeed, for $p'(x)$ =constant, (4) appears as a third degree polynomial equation in $h(x)$ whose solutions are necessarily of type $h(x)$ =constant, which is not true!

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Alternative 2: The core is at rest, i.e. $v_x = 0$ for $|y| \leq \sigma$. **THEN EITHER THERE IS NO FLOW AT ALL OR THE PROBLEM IS OVERDETERMINED! (unbalanced force acting on the plug)**

Same conclusion also for the "squeeze flow" geometry!

PARADOX!!

How to overcome the paradox? Various approaches

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- **at higher order of approximation the problematic plugs are actually above the yield stress allowing for their deformation**. These regions are termed “pseudo-plug” and their boundaries “fake yield surfaces” [Balmforth & Craster (JNNFM, 1999), Frigaard & Ryan (JNNFM, 2004), Putz, Frigaard & Martinez (JNNFM, 2009), Muravleva (2015)]. **The paradox is just a matter of the leading order approximation!**

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Classical references: Barnes et al. 1985 “The yield stress myth” , Astarita 1989 “The engineering reality of yield stress”, Möller et al. 2009 “Origin of the apparent viscosity in yield stress fluids below yielding”... and many others

Our point of view

Let's leave the "philosophical" debate about yielding/non-yielding since on the day-long time scale yield-stress fluid are a reasonable (and useful) model of real behavior and let's turn back to the main mathematical question

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Is it possible to overcome the lubrication paradox in complex geometries remaining at the (much simpler) leading order (i.e. at zeroth order in ε)?

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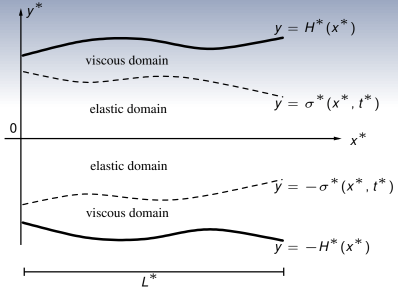
1. L. Fusi, A. Farina, F. R. *Flow of a Bingham-like fluid in a finite channel of varying width: A two scale approach* (JNNFM, 2012)
2. L. Fusi, A. Farina, F. R. *Gravity Driven Flow of a Bingham Fluid on an Inclined Surface: A Physical Paradox* (Phys. Fluids, 2012)

Horizontal channel of varying width



- $\boldsymbol{X}^*, t^*$ Lagrangian coordinates, $\boldsymbol{x}^* = \chi^*(\boldsymbol{X}^*, t^*)$ particle trajectory
- $\mathbb{F}^* = \mathrm{grad}^* \left(\chi^*(\boldsymbol{X}^*, t^*)\right)$ deformation tensor
- $\boldsymbol{u}^* = \chi^* - \boldsymbol{X}^*$ displacement, $\dfrac{\partial \boldsymbol{u}^*}{\partial t^*} = \boldsymbol{v}^*$ (lagrangian velocity)
- κ^* elasticity modulus
- $\mathbb{E}^* = \dfrac{1}{2}\left[\nabla^* \boldsymbol{u}^* + \left(\nabla^* \boldsymbol{u}^*\right)^{\mathrm{T}}\right]$ “strain” tensor
- Inlet-outlet pressures are assigned

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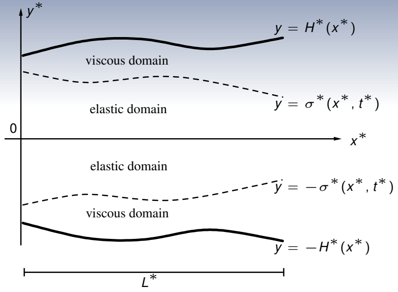

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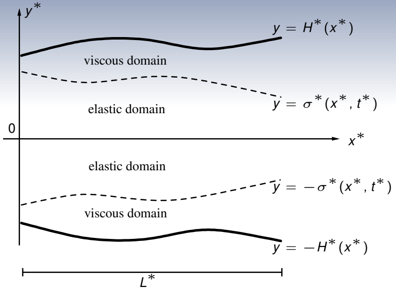

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$$\begin{cases} \mathbb{S}^* = \left(\dfrac{\tau_0^*}{II_{D^*}} + 2\eta\right)\mathbb{D}^*, & \text{when} \quad II_{S^*} > \tau_0^* \\ \underbrace{\mathbb{S}^* = 2\kappa^*\mathbb{E}^*}_{\text{linear elastic core}}, & \text{when} \quad II_{S^*} \le \tau_0^* \end{cases}$$

Dimensional scaling

$$\mathrm{Re} = \frac{\rho^* U^* h^*}{\eta^*}, \qquad y = \frac{1}{\varepsilon}\frac{y^*}{L^*}, \qquad t = \frac{t^*}{t_c^*}, \ \left(t_c^* = \frac{L^*}{U^*}\right)$$

$$v_1 = \frac{v_1^*}{U^*}, \quad v_2 = \frac{v_2^*}{\varepsilon U^*}, \quad P = \frac{P^*}{P_c^*}, \qquad H^*(x^*) = h^* h(x^*),$$

$P_c^* \overset{\mathrm{def}}{=} \dfrac{\eta^* U^* L^*}{{h^*}^2}$ (classical Poiseuille formula), $h^* \overset{\mathrm{def}}{=} \max\limits_{x^* \in [0, L^*]} H^*(x^*)$, (order of magnitude of $H^*(x^*)$) and

$$h\left(x^*\right)$$

is a dimensionless function s.t. $0 < h \leq 1$.

In our case pressure forces dominate inertia (Eu is large) and the only body force is gravity (conservative and so included in the pressure term)

Free (unknown) boundary $y = \sigma(x, t)$

The yield surface $y = \sigma(x, t)$ is NOT material!

Define the (dimensionless) Bingham number

$$\mathrm{Bi} = \frac{\tau_o^* H^*}{U^* \eta^*} \qquad \left(= \frac{\text{yield force}}{\text{viscous force}} \right)$$

Then

von Mises' criterion (dimensionless): $II_{\mathbb{S}} \lesseqgtr \mathrm{Bi}$.

- elastic region $0 < y < \sigma(x, t)$, for $II_{\mathbb{S}} < \mathrm{Bi}$;
- viscous region $\sigma(x, t) < y < h(x)$, for $II_{\mathbb{S}} \geq \mathrm{Bi}$

Dimensionless conditions on $y = \sigma(x, t)$ and $y = h(x)$

Apply Rankine-Hugoniot on the yield surface (at leading order)

$$[\![v_1]\!] = [\![v_2]\!] = 0, \qquad [\![P]\!] = 0, \qquad [\![S_{12}]\!] = 0 \tag{5}$$

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Over $y = h(x)$ (rigid)

$$\boldsymbol{v}\left(x, h\left(x\right), t\right) \equiv 0, \quad \text{(impermeable, no-slip)}$$

Simmetry on $y = 0$ for the displacement $\boldsymbol{u}$

$$\left.\frac{\partial u_1}{\partial y}\right|_{y=0} = 0, \qquad u_2\left(x, 0, t\right) = 0, \qquad \left.\frac{\partial u_2}{\partial y}\right|_{y=0} = 0.$$

Elastic domain: $0 < x < 1,\ 0 < y < \sigma(x,t)$

Define

$$\Gamma \overset{\text{def}}{=} \frac{\kappa^* h^*}{\eta^* U^*} = \text{Re}\left(\frac{U^*}{c^*}\right)^{-2}$$

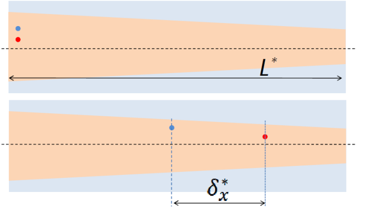
being $c^* \overset{\text{def}}{=} \sqrt{\kappa^*/\rho^*}$ the sound velocity.

Dimensionless form of the shear stress in the unyielded region

$$\mathbb{S} = 2\Gamma\,\mathbb{E} = \Gamma \begin{pmatrix} 2\dfrac{\partial u_1}{\partial x} & \dfrac{1}{\varepsilon}\dfrac{\partial u_1}{\partial y} + \varepsilon\dfrac{\partial u_1}{\partial x} \\ \dfrac{1}{\varepsilon}\dfrac{\partial u_1}{\partial y} + \varepsilon\dfrac{\partial u_1}{\partial x} & 2\dfrac{\partial u_1}{\partial y} \end{pmatrix}$$

Γ is a crucial parameter!

Core deformation



Physical meaning of Γ

The parameter Γ gives a measure of the deformation of the core

$$\text{strain} = \frac{\delta_x^*}{L^*}, \qquad \text{strain} = \frac{\text{stress}}{\kappa^*} = \frac{\eta^* U^* / L^*}{\kappa^*} = \left(\frac{\eta^* U^*}{\kappa^* h^*}\right)\left(\frac{h^*}{L^*}\right) = \frac{\varepsilon}{\Gamma}$$

Core deformation

The core deformation is “visible” if the strain is $\mathcal{O}(1)$

$$\text{strain} = \mathcal{O}(1) \Leftrightarrow \frac{\varepsilon}{\Gamma} = \mathcal{O}(1)) \Leftrightarrow \Gamma = \mathcal{O}(\varepsilon)$$

$\Gamma = \mathcal{O}(\varepsilon)$ corresponds to inner core **non-negligible** deformations. Such a case is totally different from the classical Bingham model. $\Gamma = \mathcal{O}(1)$ corresponds to inner core negligible deformations.

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Theorem (generalization of Lipscomb-Denn's remark)

In a channel with non-constant width $h(x)$ and $\Gamma = \mathcal{O}(1)$ (elastic core with negligible deformations) the lubrication approximation at the leading order leads to a contradiction and the classical paradox is retrieved.

L. Fusi, A. Farina, F. R. *Flow of a Bingham-like fluid in a finite channel of varying width: A two scale approach* (JNNFM, 2012)

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However the case $\Gamma = \mathcal{O}(\varepsilon)$ is different and no paradox arises!

Case $\Gamma = \mathcal{O}(\varepsilon)$

At the leading order $P^{(0)} = P^{(0)}(x,t)$, in the whole domain.

In the core

$$\left\{ \begin{array}{l} \dfrac{\partial^2 u_1^{(0)}}{\partial y^2} = \dfrac{\varepsilon}{\Gamma}\dfrac{\partial P^{(0)}}{\partial x}, \\[2ex] \left.\dfrac{\partial u_1^{(0)}}{\partial y}\right|_{y=0} = 0, \end{array} \right. \quad \Rightarrow \quad \underbrace{u_1^{(0)} = \frac{\varepsilon}{2\,\Gamma}\frac{\partial P^{(0)}}{\partial x}y^2 + a(x,t)}_{\text{elastic displacement}} \tag{6}$$

Longitudinal velocity in the core $v_1^{(0)} = \dfrac{\varepsilon}{2\,\Gamma}\dfrac{\partial^2 P^{(0)}}{\partial t \partial x}y^2 + \omega(x,t)$ with $\omega(x,t) \overset{\text{def}}{=} \dfrac{\partial a(x,t)}{\partial t}$

$\omega(x,t)$ can be interpreted as the "uniform" part (i.e. independent of y) of the longitudinal velocity of the core. The other term is a non-uniform displacement (namely a deformation) modulated by the pressure gradient. The latter term may become negligible in case $\Gamma = \mathcal{O}(1)$

Case $\Gamma = \mathcal{O}(\varepsilon)$

When we focus on the **fluid region** ($\sigma < y \leq h$) we get

$$\frac{\partial P^{(0)}}{\partial x}\sigma = -\mathrm{Bi},$$

and

$$v_1^{(0)}(x, y, t) = -\frac{1}{2}\frac{\partial P^{(0)}}{\partial x}\left(h^2 - y^2\right) + \frac{\partial P^{(0)}}{\partial x}\sigma\,(h - y)\,.$$

Next apply

1. continuity of the tangential stress at the interface σ
2. continuity of the longitudinal component of the velocity at σ
3. mass conservation

Case $\Gamma = \mathcal{O}(\varepsilon)$

Mathematical problem at the leading order ($P^{(0)}, \sigma, \omega$ unknowns)

$$(7) \qquad \begin{cases} \dfrac{\partial P^{(0)}}{\partial x}\sigma = -\mathrm{Bi}, \\[2ex] \dfrac{1}{2\,\Gamma}\dfrac{\partial^2 P^{(0)}}{\partial t \partial x}\sigma^2 + \omega\,(x,t) = -\dfrac{1}{2}\dfrac{\partial P^{(0)}}{\partial x}\,(h-\sigma)^2, \\[2ex] -\,(h-\sigma)^2\left[\dfrac{1}{6}\dfrac{\partial^2 P^{(0)}}{\partial x^2}\,(2h+\sigma) + \dfrac{\partial P^{(0)}}{\partial x}\dfrac{\partial h}{\partial x}\right] + \dfrac{\sigma^3}{6\,\Gamma}\dfrac{\partial}{\partial t}\left(\dfrac{\partial^2 P^{(0)}}{\partial x^2}\right) + \sigma\dfrac{\partial \omega}{\partial x} = 0. \end{cases}$$

Theorem

The stationary version of problem (7) admits a unique solution provided that

$$(8) \qquad \frac{\Delta P}{\mathrm{Bi}} > \int_0^1 \frac{dx}{h\,(x)}$$

Case $\Gamma = \mathcal{O}(\varepsilon)$

Condition $\boxed{\dfrac{\Delta P}{\mathrm{Bi}} > \displaystyle\int_0^1 \frac{dx}{h(x)}}$ generalizes the classical *flow condition* which holds for a "flat" channel

Actually, using physical dimensions, (8) becomes

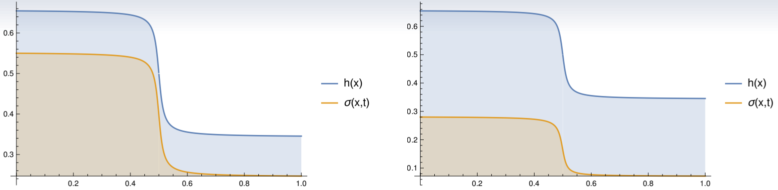
$$\frac{\Delta P^*}{\tau_o^*} > \int_0^{L^*} \frac{dx^*}{H^*(x^*)} = \frac{L^*}{h^*} \underbrace{\int_0^1 \frac{dx}{h(x)}}_{\mathcal{A}}$$

while the classical Bingham flow condition is $\dfrac{\Delta P^*}{\tau_o^*} > \dfrac{L^*}{h^*}$ or, dimensionless, $\boxed{\dfrac{\Delta P}{\mathrm{Bi}} > \dfrac{1}{h}}$

Good question: are there physical situations in which $\Gamma = O(\varepsilon)$? In our paper we mention some cases of physical interest (for example some oil-in-water emulsions).

Horizontal channel of varying width

Example: $h(x) = \frac{1}{2} - \frac{1}{10} \arctan\left[80\left(x - \frac{1}{2}\right)\right] \quad \left(\mathcal{A} = \int_0^1 \frac{1}{h} = 2.18\right)$



$\Delta P/\mathrm{Bi} = 3$ (left) $\qquad \Delta P/\mathrm{Bi} = 8$ (right)

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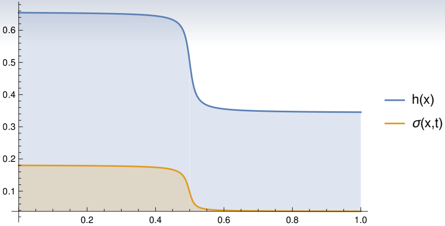


$\Delta P/\mathrm{Bi} = 15$

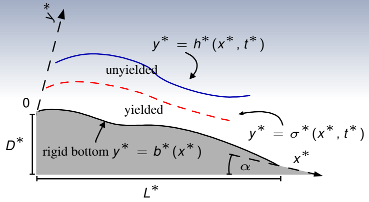
In any case the amplitude of the inner core decreases as $\Delta P/\mathrm{Bi}$ increases. For $\Delta P/\mathrm{Bi} \gg 1$ the inner core tends to disappear and the system to a purely viscous behavior. Similarly as $\Delta P/\mathrm{Bi} \to \mathcal{A}$, then $\sigma \to h$ and the viscous layer tends to disappear.

One open problem

Existence and uniqueness for the **non-stationary** problem at the zeroth order in ε (with the appropriate BCs and ICs)

$$\begin{cases} \dfrac{\partial P^{(0)}}{\partial x}\sigma = -\mathrm{Bi}, \\[2ex] \dfrac{1}{2\,\Gamma}\dfrac{\partial^2 P^{(0)}}{\partial t \partial x}\sigma^2 + \omega\left(x,t\right) = -\dfrac{1}{2}\dfrac{\partial P^{(0)}}{\partial x}\left(h-\sigma\right)^2, \\[2ex] -\left(h-\sigma\right)^2\left[\dfrac{1}{6}\dfrac{\partial^2 P^{(0)}}{\partial x^2}\left(2h+\sigma\right) + \dfrac{\partial P^{(0)}}{\partial x}\dfrac{\partial h}{\partial x}\right] + \quad \dfrac{\sigma^3}{6\,\Gamma}\dfrac{\partial}{\partial t}\left(\dfrac{\partial^2 P^{(0)}}{\partial x^2}\right) + \sigma\dfrac{\partial \omega}{\partial x} = 0. \end{cases}$$

Gravity motion of a Bingham fluid: a new paradox



Hypotheses:

- $b^*(x^*)$ sufficiently regular, $(b^*)'$ bounded
- $\tan\alpha = O(1)$ or $\tan\alpha = O(\varepsilon)$
- $\mathrm{Re} = \mathcal{O}(\varepsilon)$,
- lubrication hypothesis applies (thin film)

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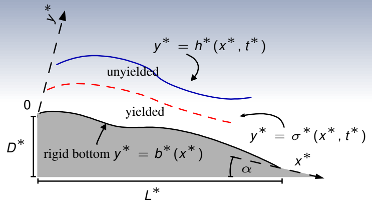

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Use the classical Bingham (rigid core) model

$$\begin{cases} \mathbb{S}^* = \left(\dfrac{\tau_0^*}{II_{D^*}} + 2\eta\right)\mathbb{D}^*, & \text{when} \quad II_{S^*} > \tau_0^* \\ \mathbb{D}^* = \mathbb{O} & \text{when} \quad II_{S^*} \leq \tau_0^* \end{cases}$$

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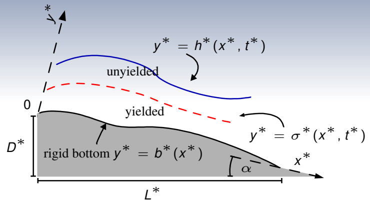

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Various inconsistencies appear again. This case was never considered in the literature! The failure of the model is proved to be of strict mathematical nature (ill-posedness)

Gravity motion of a Bingham fluid: a new paradox

The mathematical (hyperbolic) problem

$$\begin{cases} 6\dfrac{\partial(\sigma-b)}{\partial t}=-\dfrac{\partial(\sigma-b)^3}{\partial x}, & x\in(0,1) \quad t>0 \\[2ex] \dfrac{1}{3}\sigma(0,t)+\dfrac{\mathrm{Bi}}{2}\sigma^2(0,t)=q(t) & x=1, \quad\quad t>0 \\[2ex] +\mathrm{I.C.} \end{cases}$$

$$q(t)=\int_0^{h(0,t)} u_1(0,y,t)\,dy \quad \text{(discharge)}$$

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Proposition

A necessary condition for the mathematical problem to have a **physically meaningful solution** is that, within the solid domain $[\sigma, h]$ the displacement $\boldsymbol{u}$ fulfill the following conditions

$$\begin{cases} u_1 = \kappa_1(t)-\varepsilon^2\omega(t)y, \\ u_2 = \omega(t)x+\kappa_2(t), \\ \text{with} \quad \kappa_1=\mathcal{O}(1),\ \kappa_2=\mathcal{O}(1),\ \omega=\mathcal{O}(1). \end{cases}$$

Gravity motion of a Bingham fluid: a new paradox!

Theorem 1

If the mathematical problem for the unknown yield surface $\sigma(x,t)$ admits “physically meaningful” solutions, then the shear stress $\mathbb{S}$ **remains bounded** both at the leading and first order, as $y \to \sigma$

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L. Fusi, A. Farina, F. R. On the mathematical paradoxes for the flow of a viscoplastic film down an inclined surface (JNLM, 2014)

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All standard approaches follow this procedure: the dimensional shear stress $\mathbb{S}^*$ is re-scaled with the characteristic viscous stress, **tacitly assuming that the dimensionless stress $\mathbb{S}$ is everywhere** $\mathcal{O}(1)$. Then one uses the momentum balance “in its local form”

$$\varrho\left(\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}\right)=-\nabla p+\nabla\cdot\mathbb{S}$$

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We proposed a new strategy

L. Fusi, A. Farina, F. R. Pressure driven lubrication flow of a Bingham fluid in a channel: a novel approach (JNNFM, 2015)

Our approach

PROBLEM: in the rigid core the stress is not determined!

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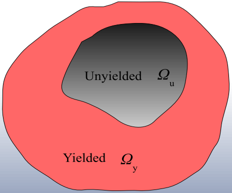
ANSWER

Use a global integral approach to model the unyielded part.



(**Classical approach**)

$$\boxed{\varrho\left(\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}\right)=-\nabla p+\nabla\cdot\mathbb{S}\quad\text{in}\quad\Omega_u\cup\Omega_y}$$

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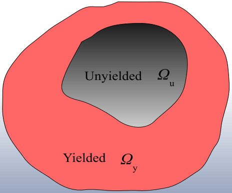

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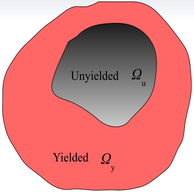
(**Novel approach**)

$$\boxed{\varrho\left(\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}\right)=-\nabla p+\nabla\cdot\mathbb{S}\quad\text{in}\quad\Omega_y}$$

The main step



Recall: **the domain Ω_u is a non-material volume whose boundary $\partial\Omega_u$ moves with velocity $\boldsymbol{w}$**

Integral form of the momentum balance

$$\frac{d}{dt}\int_{\Omega_u}\rho\boldsymbol{v}d\boldsymbol{x} = \int_{\partial\Omega_u}\mathbb{T}\boldsymbol{n}d\sigma - \underbrace{\int_{\partial\Omega_u}\rho\boldsymbol{v}\Big[(\boldsymbol{v}-\boldsymbol{w})\cdot\boldsymbol{n}\Big]d\sigma}_{\text{extra term}}.$$

The “extra term” is due to the non-material nature of Ω_u.

Apply Reynolds transport theorem

$$\frac{d}{dt}\int_{\Omega_u}\rho\boldsymbol{v}d\boldsymbol{x}=\int_{\Omega_u}\frac{\partial}{\partial t}\Big(\rho\boldsymbol{v}\Big)d\boldsymbol{x}+\int_{\partial\Omega_u}\rho\boldsymbol{v}\Big(\boldsymbol{w}\cdot\boldsymbol{n}\Big)d\sigma.$$

Integral momentum balance of the unyielded part

$$\int_{\Omega_u}\frac{\partial}{\partial t}\Big(\rho\boldsymbol{v}\Big)d\boldsymbol{x}+\int_{\partial\Omega_u}\rho\boldsymbol{v}\Big(\boldsymbol{v}\cdot\boldsymbol{n}\Big)d\sigma=\int_{\partial\Omega_u}\mathbb{T}\boldsymbol{n}d\sigma$$

Final set of equations to solve

- $\varrho\left(\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}\right)=-\nabla p+\nabla\cdot\mathbb{S}$ in Ω_y **(yielded part)**
- $\int\limits_{\Omega_u}\frac{\partial}{\partial t}\left(\rho\boldsymbol{v}\right)d\boldsymbol{x}+\int\limits_{\partial\Omega_u}\rho\boldsymbol{v}\left(\boldsymbol{v}\cdot\boldsymbol{n}\right)d\sigma=\int\limits_{\partial\Omega_u}\mathbb{T}\boldsymbol{n}d\sigma$ in Ω_u **(unyielded part)**
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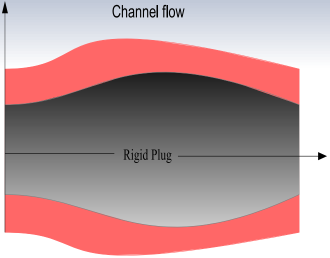
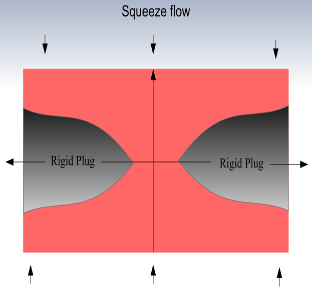
The advantage of using the integral form in Ω_u lies in the fact that we only need to know the stress $\mathbb{T}$ acting on $\partial\Omega_u$ exerted by the yielded part (i.e. the viscous stress).

2D Bingham Models Studied





(1) Channel flow with non uniform walls

(2) Planar squeeze flow

Example 1: Channel flow at the leading order with the new approach

Manipulations of the system lead to an integro-differential equation for the pressure:

$$\frac{\partial^2 p}{\partial x^2} + \frac{6\frac{\mathrm{d}\, h}{\mathrm{d}\, x}}{\left[3h + \frac{2\int_0^1 \frac{\partial p}{\partial x} h\, dx - \mathrm{Bi}}{\Delta P}\right]} \frac{\partial p}{\partial x} = 0, \qquad p(0,t) = \Delta p, \quad p(1,t) = 0$$

Once p is found, we use it to determine v_1, v_2 and the yield surface σ and the problem is completely solved

In particular

$$\sigma(x,t) = 2(h(0) - h(x)) + \frac{\mathrm{Bi}}{\Delta P(t)} + \frac{2}{\Delta P(t)} \int_0^1 p(x,t) \frac{\mathrm{d}\, h}{\mathrm{d}\, x}$$

REMARK

Notice that $\sigma_x = -2h_x$, i.e. the core amplitude widens as the channel narrows, whereas it shrinks as the channel becomes wider. When h is flat, then also σ is flat and we recover the classical 1D Bingham flow.

Example 1: Channel flow at the leading order with the new approach

It is important to determine a condition upon the physical parameters that ensure that the system is actually flowing):

Flow condition

Define

$$\mathfrak{h} = 3h_{\min} - 2\left\|h_x\right\|_{L^2} - 2h_{in}$$

and assuming $\mathfrak{h} > 0$ the flow condition becomes

$$\Delta p > \frac{\mathrm{Bn}}{\mathfrak{h}}.$$

Remark

When h is constant we recover the flow condition of the classical 1D Bingham model ($\mathfrak{h} = h$).

Channel flow simulations

$$h(x) = \frac{\arctan(5(1-2x)/2)}{4\arctan(5/2)} + \frac{3}{4}$$

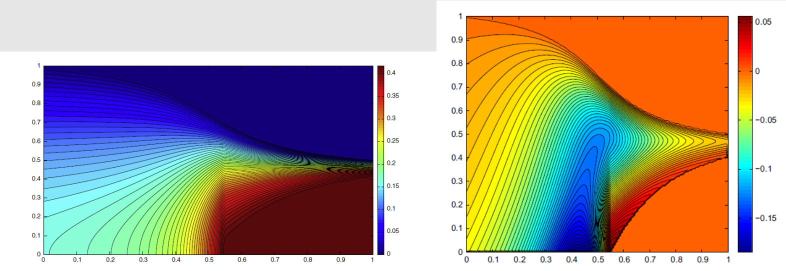


Figure: Velocity components v_1 (longitudinal, on the center), and v_2 (transversal, on the right). Solid colors means uniformity, h and σ are clearly visible

Channel flow simulations

$$h(x) = 1 - (3/10)\cos\left(\pi\left(x - \frac{1}{2}\right)\right)$$

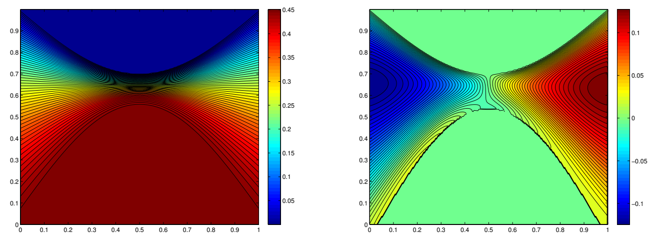
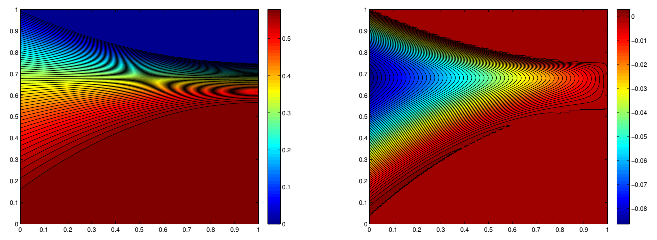


Figure: Wall and σ profile, velocity components v_1 (longitudinal, on the center), and v_2 (transversal, on the right). Solid colors means uniformity, h and σ are clearly visible

Channel flow simulations

$$h(x) = h(x) = ((x-1)^2 + 3)/4$$



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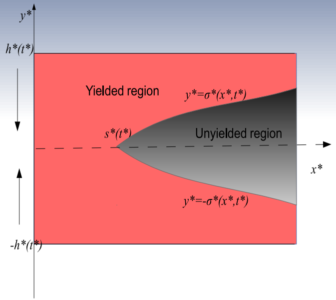
Planar squeeze geometry: sketch of the system



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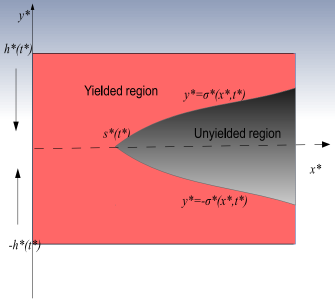

- The fluid is confined between parallel plates moving with velocity $\dot{h}^*(t)$

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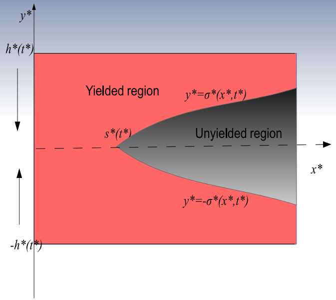

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- The fluid is squeezed out of the gap. The unyielded part is detached from the wall $x^* = 0$

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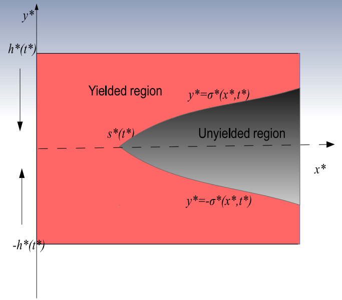

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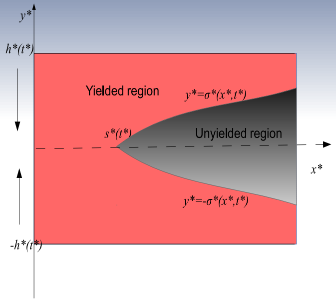

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Bingham constitutive equation

$$\mathbb{T}^* = -p^*\mathbb{I} + \mathbb{S}^*$$

$$\mathbb{S}^* = \left(2\mu + \frac{\tau_o^*}{II_D^*}\right)\mathbb{D} \qquad II_S^* \geqslant \tau_o^*$$

$$\mathbb{D}^* = 0 \qquad II_S^* \leqslant \tau_o^*$$

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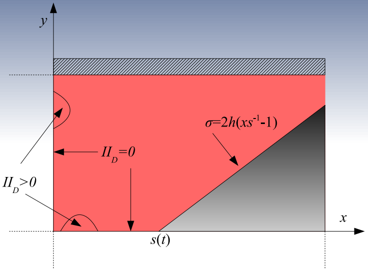

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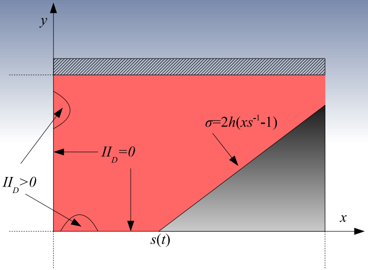

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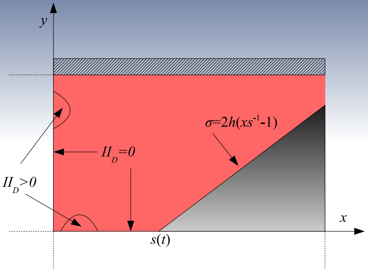

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The axis $x = 0$ and the segment $\{y = 0,\ 0 \leq x \leq s(t)\}$ are unyielded regions with zero measure!

They remain “invisible” at leading order but “visible” at higher order.

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Theorem

- **In a neighbourhood of the closed end the material is always yielded so that the rigid part is always detached from it. For planar rigid walls the solution is "almost" explicit.**
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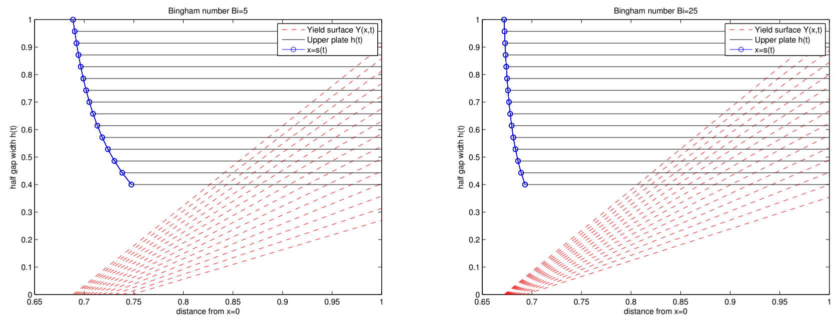
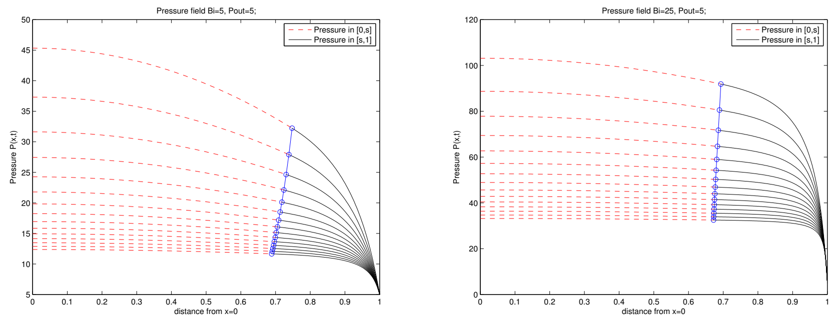
Planar case with rigid unyielded domain

L. Fusi, A. Farina, F. R., *Planar squeeze flow of a Bingham fluid*, Journal of Non-Newtonian Fluid Mechanics 225, (2015), 1–9.

Yield surface plot: $h(t) = 1 - t, \qquad \dot{h}(t) = -1$



Pressure plot: $h(t) = 1 - t, \qquad \dot{h}(t) = -1$



Non flat plates: $h = h(x, t)$

Yield surface

$$\sigma(x, t) = \max\left\{0; -2h(x, t) + 2h(s, t)\frac{\int_0^x h_t dx'}{\int_0^s h_t dx'}\right\}$$

Equation for $s(t)$

$$\int_s^1 \frac{\left[-2h|_x + 2h|_s \frac{\int_0^x h_t dx'}{\int_0^s h_t dx'}\right]_+}{\left[h|_x - \left[-2h|_x + 2h|_s \frac{\int_0^x h_t dx'}{\int_0^s h_t dx'}\right]_+\right]^2} dx = -\frac{\mathrm{Bi}}{\frac{3}{h|_s}\int_0^s h_t dx'}$$

Axisymmetric case (schematic picture)

For the axisymmetric case the rigid core assumption leads **unavoidably** to a paradox.

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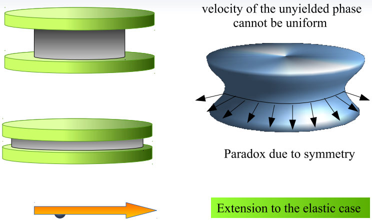
For the axisymmetric case the rigid core assumption leads **unavoidably** to a paradox. However ...The hypothesis of a linearly elastic core again allows to overcome the lubrication paradox!



velocity of the unyielded phase
cannot be uniform

Paradox due to symmetry

Extension to the elastic case

Axisymmetric case with linear elastic unyielded domain

L. Fusi, A. Farina, F. R., *Squeeze Flow of a Bingham-Type Fluid with Elastic core* , Int. Journal of Nonlinear Mechanics, 78, (2016), 59–65.

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6. Other more complex geometries are important in applications: an exhaustive analysis is still missing

Muchas gracias por su amable atención